Option syllabus content

Topic 8—Option: Statistics and probability 40 hrs

Aims

The aims of this option are to allow students the opportunity to approach statistics in a practical way; to demonstrate a good level of statistical understanding; and to understand which situations apply and to interpret the given results. It is expected that GDCs will be used throughout this option and that the minimum requirement of a GDC will be to find pdf, cdf, inverse cdf, *p*-values and test statistics including calculations for the following distributions: binomial, Poisson, normal, *t* and chi-squared. Students are expected to set up the problem mathematically and then read the answers from the GDC, indicating this within their written answers. Calculator-specific or brand-specific language should not be used within these explanations.

Details

	Content	Amplifications/inclusions	Exclusions
8.1	Expectation algebra.	$\mathrm{E}(aX+b)=a\mathrm{E}(X)+b$; $\mathrm{Var}(aX+b)=a^2\mathrm{Var}(X)$.	
	Linear transformation of a single random variable.		
	Mean and variance of linear combinations of two independent random variables. Extension to linear combinations of *n* independent random variables.	$\mathrm{E}(a_1X_1 \pm a_2X_2)=a_1\mathrm{E}(X_1)\pm a_2\mathrm{E}(X_2)$; $\mathrm{Var}(a_1X_1 \pm a_2X_2)=a_1^2\mathrm{Var}(X_1)+a_2^2\mathrm{Var}(X_2)$.	

Topic 8—Option: Statistics and probability (continued)

	Content	Amplifications/inclusions	Exclusions
8.2	Cumulative distribution functions.		Formal treatment of proof of means and variances.
	Discrete distributions: uniform, Bernoulli, binomial, negative binomial, Poisson, geometric, hypergeometric.	Probability mass functions, means and variances.	
	Continuous distributions: uniform, exponential, normal.	Probability density functions, means and variances.	
8.3	Distribution of the sample mean.		Sampling without replacement.
	The distribution of linear combinations of independent normal random variables. In particular $X \sim \mathrm{N}(\mu, \sigma^2) \Rightarrow \bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$.	A linear combination of independent normally distributed random variables is also normally distributed.	
	The central limit theorem.		Proof of the central limit theorem.
	The approximate normality of the proportion of successes in a large sample.	The extension of these results for large samples to distributions that are not normal, using the central limit theorem.	Distributions that do not satisfy the central limit theorem.

Topic 8—Option: Statistics and probability (continued)

	Content	Amplifications/inclusions	Exclusions
8.4	Finding confidence intervals for the mean of a population. Finding confidence intervals for the proportion of successes in a population.	Use of the normal distribution when σ is known and the *t*-distribution when σ is unknown (regardless of sample size). The case of paired samples (matched pairs) could be tested as an example of a single sample technique.	The difference of means and the difference of proportions.
8.5	Significance testing for a mean. Significance testing for a proportion. Null and alternative hypotheses H_0 and H_1. Type I and Type II errors. Significance levels; critical region, critical values, *p*-values; one-tailed and two-tailed tests.	Use of the normal distribution when σ is known and the *t*-distribution when σ is unknown. The case of paired samples (matched pairs) could be tested as an example of a single sample technique.	The difference of means and the difference of proportions.

Topic 8—Option: Statistics and probability (continued)

	Content	Amplifications/inclusions	Exclusions
8.6	The chi-squared distribution: degrees of freedom, υ.		
	The χ^2 statistic, $\chi^2_{calc} = \sum \frac{(f_o - f_e)^2}{f_e}$.	Awareness of the fact that χ^2_{calc} is a measure of the discrepancy between observed and expected values.	
	The χ^2 goodness of fit test.	Test for goodness of fit for all of the above distributions; the requirement to combine classes with expected frequencies of less than 5.	
	Contingency tables: the χ^2 test for the independence of two variables.		Yates' continuity correction for $\upsilon = 1$.

Topic 9—Option: Sets, relations and groups

40 hrs

Aims

The aims of this option are to provide the opportunity to study some important mathematical concepts, and introduce the principles of proof through abstract algebra.

Details

	Content	Amplifications/inclusions	Exclusions
9.1	Finite and infinite sets. Subsets. Operations on sets: union; intersection; complement, set difference, symmetric difference.		
	De Morgan's laws; distributive, associative and commutative laws (for union and intersection).	Illustration of these laws using Venn diagrams.	Proofs of these laws.
9.2	Ordered pairs: the Cartesian product of two sets.		
	Relations; equivalence relations; equivalence classes.	An equivalence relation on a set induces a partition of the set.	
9.3	Functions: injections; surjections; bijections.	The term "codomain".	
	Composition of functions and inverse functions.	Knowledge that the function composition is not a commutative operation and that if f is a bijection from set A onto set B then f^{-1} exists and is a bijection from set B onto set A.	

Topic 9—Option: Sets, relations and groups (continued)

	Content	Amplifications/inclusions	Exclusions
9.4	Binary operations.	A binary operation $*$ on a non-empty set S is a rule for combining any two elements $a,b \in S$ to give a unique element c. That is, in this definition, a binary operation is not necessarily closed. On examination papers: candidates may be required to test whether a given operation satisfies the closure condition.	
	Operation tables (Cayley tables).	Operation tables with the Latin square property (every element appears once only in each row and each column).	
9.5	Binary operations with associative, distributive and commutative properties.	The arithmetic operations in $\mathbb{R}$ and $\mathbb{C}$; matrix operations.	
9.6	The identity element e.	Both the right-identity $a * e = a$ and left-identity $e * a = a$ must hold if e is an identity element.	
	The inverse a^{-1} of an element a.	Both $a * a^{-1} = e$ and $a^{-1} * a = e$ must hold.	
	Proof that left-cancellation and right-cancellation by an element a hold, provided that a has an inverse.		
	Proofs of the uniqueness of the identity and inverse elements.		

Topic 9—Option: Sets, relations and groups (continued)

	Content	Amplifications/inclusions	Exclusions
9.7	The axioms of a group $\{G, *\}$.	For the set G under a given operation $*$: • G is closed under $*$ • $*$ is associative • G contains an identity element • each element in G has an inverse in G.	
	Abelian groups.	$a * b = b * a$, for all $a, b \in G$.	
9.8	The groups: • $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{C}$ under addition • matrices of the same order under addition • 2×2 invertible matrices under multiplication • integers under addition modulo n		
	• groups of transformations	The composition $T_1 T_2$ denotes T_2 followed by T_1.	
	• symmetries of an equilateral triangle, rectangle and square • invertible functions under composition of functions		
	• permutations under composition of permutations.	On examination papers: the form $p = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ will be used to represent the mapping $1 \rightarrow 3$, $2 \rightarrow 1$, $3 \rightarrow 2$.	

Topic 9—Option: Sets, relations and groups (continued)

	Content	Amplifications/inclusions	Exclusions
9.9	Finite and infinite groups. The order of a group element and the order of a group.	Latin square property of a group table.	
9.10	Cyclic groups. Proof that all cyclic groups are Abelian.	Generators.	
9.11	Subgroups, proper subgroups. Use and proof of subgroup tests.	Suppose G is a group and H is a non-empty subset of G. H is a subgroup of G if $ab^{-1} \in H$ whenever $a, b \in H$. Suppose G is a finite group and H is a non-empty subset of G. H is a subgroup of G if H is closed under the group operation.	
	Lagrange's theorem. Use and proof of the result that the order of a finite group is divisible by the order of any element. (Corollary to Lagrange's theorem.)		On examination papers: questions requiring the proof of Lagrange's theorem will not be set.

Topic 9—Option: Sets, relations and groups (continued)

	Content	Amplifications/inclusions	Exclusions
9.12	Isomorphism of groups.	Infinite groups as well as finite groups. Two groups $\{G, \circ\}$ and $\{H, \bullet\}$ are isomorphic if there exists a bijection $f: G \to H$ such that $f(a \circ b) = f(a) \bullet f(b)$ for all $a, b \in G$. The function $f: G \to H$ is an isomorphism.	
	Proof of isomorphism properties for identities and inverses.	Identity: let e_1 and e_2 be the identity elements of G, H respectively, then $f(e_1) = e_2$. Inverse: $f(a^{-1}) = (f(a))^{-1}$ for all $a \in G$.	

Topic 10—Option: Series and differential equations 40 hrs

Aims

The aims of this option are to introduce limit theorems and convergence of series, and to use calculus results to solve differential equations.

Details

	Content	Amplifications/inclusions	Exclusions
10.1	Infinite sequences of real numbers.		
	Limit theorems as *n* approaches infinity.	Limit of sum, difference, product, quotient; squeeze theorem.	
	Limit of a sequence.	Formal definition: the sequence $\{u_n\}$ converges to the limit *L*, if for any $\varepsilon > 0$, there is a positive integer *N* such that $\lvert u_n - L \rvert < \varepsilon$, for all $n > N$.	
	Improper integrals of the type $\int_a^\infty f(x)\,\mathrm{d}x$.		
	The integral as a limit of a sum; lower sum and upper sum.		

Topic 10—Option: Series and differential equations (continued)

	Content	Amplifications/inclusions	Exclusions
10.2	Convergence of infinite series.	The sum of a series is the limit of the sequence of its partial sums.	
	Partial fractions and telescoping series (method of differences).	Simple linear non-repeated denominators.	
	Tests for convergence: comparison test; limit comparison test; ratio test; integral test.	Students should be aware that if $\lim_{x\to\infty} x_n = 0$ then the series is not necessarily convergent, but if $\lim_{x\to\infty} x_n \neq 0$, the series diverges.	
	The *p*-series, $\sum \frac{1}{n^p}$.	$\sum \frac{1}{n^p}$ is convergent for $p > 1$ and divergent otherwise. When $p = 1$, this is the harmonic series.	
	Use of integrals to estimate sums of series.		
10.3	Series that converge absolutely.		
	Series that converge conditionally.		
	Alternating series.	Conditions for convergence. The absolute value of the truncation error is less than the next term in the series.	
10.4	Power series: radius of convergence and interval of convergence. Determination of the radius of convergence by the ratio test.		

Topic 10—Option: Series and differential equations (continued)

	Content	Amplifications/inclusions	Exclusions
10.5	Taylor polynomials and series, including the error term.	Applications to the approximation of functions; formulae for the error term, both in terms of the value of the $(n+1)^{\text{th}}$ derivative at an intermediate point, and in terms of an integral of the $(n+1)^{\text{th}}$ derivative.	Proof of Taylor's theorem.
		Differentiation and integration of series (valid only on the interval of convergence of the initial series).	Use of products and quotients to obtain other series.
	Maclaurin series for e^x, $\sin x$, $\cos x$, $\arctan x$, $\ln(1+x)$, $(1+x)^p$. Use of substitution to obtain other series.	Intervals of convergence for these Maclaurin series. Example: e^{x^2}.	
	The evaluation of limits of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ using l'Hôpital's Rule and/or the Taylor series.	Cases where the derivatives of $f(x)$ and $g(x)$ vanish for $x = a$.	Proof of l'Hôpital's Rule.

Topic 10—Option: Series and differential equations (continued)

	Content	Amplifications/inclusions	Exclusions
10.6	First order differential equations: geometric interpretation using slope fields; numerical solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ using Euler's method. Homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$ using the substitution $y = vx$. Solution of $y' + P(x)y = Q(x)$, using the integrating factor.	 $y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$, where h is a constant.	

Topic 11—Option: Discrete mathematics 40 hrs

Aims

The aim of this option is to provide the opportunity for students to engage in logical reasoning, algorithmic thinking and applications.

Details

	Content	Amplifications/inclusions	Exclusions
11.1	Division and Euclidean algorithms.	The theorem $a \mid b$ and $a \mid c \Rightarrow a \mid (bx \pm cy)$ where $x, y \in \mathbb{Z}$. The division algorithm $a = bq + r$, $0 \le r < b$.	
	The greatest common divisor, $\gcd(a,b)$, and the least common multiple, $\text{lcm}(a,b)$, of integers a and b.	The Euclidean algorithm for determining the greatest common divisor of two integers.	
	Relatively prime numbers; prime numbers and the fundamental theorem of arithmetic.		Proof of the fundamental theorem of arithmetic.
11.2	Representation of integers in different bases.	On examination papers: questions that go beyond base 16 are unlikely to be set.	
11.3	Linear diophantine equations $ax + by = c$.	General solutions required and solutions subject to constraints. For example, all solutions must be positive.	
11.4	Modular arithmetic. Linear congruences. Chinese remainder theorem.		
11.5	Fermat's little theorem.	$a^p \equiv a \pmod p$ where p is prime.	On examination papers: questions requiring proof of the theorem will not be set.

Topic 11—Option: Discrete mathematics (continued)

	Content	Amplifications/inclusions	Exclusions
11.6	Graphs, vertices, edges. Adjacent vertices, adjacent edges.	Two vertices are adjacent if they are joined by an edge. Two edges are adjacent if they have a common vertex.	
	Simple graphs; connected graphs; complete graphs; bipartite graphs; planar graphs, trees, weighted graphs.	Euler's relation: $v-e+f=2$; theorems for planar graphs including $e \le 3v-6$, $e \le 2v-4$, κ_5 and $\kappa_{3,3}$ are not planar.	
	Subgraphs; complements of graphs.		
	Graph isomorphism.	Simple graphs only for isomorphism.	
11.7	Walks, trails, paths, circuits, cycles.	A connected graph contains a Eulerian circuit if and only if every vertex of the graph is of even degree.	Dirac's theorem for Hamiltonian cycles.
	Hamiltonian paths and cycles; Eulerian trails and circuits.		
11.8	Adjacency matrix.	Applications to isomorphism and of the powers of the adjacency matrix to number of walks.	
	Cost adjacency matrix.		
11.9	Graph algorithms: Prim's; Kruskal's; Dijkstra's.	These are examples of "greedy" algorithms.	

Topic 11—Option: Discrete mathematics (continued)

	Content	Amplifications/inclusions	Exclusions
11.10	“Chinese postman” problem (“route inspection”).	To determine the shortest route around a weighted graph going along each edge at least once (route inspection algorithm).	Graphs with more than two vertices of odd degree.
	“Travelling salesman” problem.	To determine the Hamiltonian cycle of least weight in a weighted complete graph.	
	Algorithms for determining upper and lower bounds of the travelling salesman problem.		Graphs in which the triangle inequality is not satisfied.